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Construction of control chart using process capability for average system length in $M^{[x]}/M/1$ Queueing system

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Abstract: **In queueing theory, a discipline within the mathematical theory of probability, a bulk queue is a general queueing model where jobs arrive in and/or are served in groups of random size. Queueing models characterized by bulk arrival or bulk service of either fixed or variable sizes are commonly found in modelling of traffic and transportation systems, complex computer and telecommunication systems, inventory replenishment system and other real-life applications. Control chart technique may be applied to analyse the waiting time of the customers in the system to improve the services and the effective performance of concerns.In this research article the construction of control chart using process capability for average system length is proposed and provides suitable tables with numerical results for $M^{[x]}/M/1$ queueing systemwhen the batch sizefollows geometric distribution.**

Keywords: **Average run length, Exponential service time, Geometric distribution, Poisson arrival, Process capability.**

I. INTRODUCTION

Waiting in line for service is the most unpleasant experiences in this world. Barrer (1957) says, in queuing processes a potential customer is considered "lost" if the system is busy at the time service is demanded. If not served during this time, the customer leaves the system and is considered lost. Some of the examples of queueing situations are in which customers arrive in groups such as'Letters arriving at a post office', 'ships arriving at a port in a convoy', 'people attending a wedding reception' etc., Queueing model consisting of bulkarrival has been discussed by Gross and Harris (1998). In queuing system the customer satisfaction can be increased by constructing control charts for average system length and providing control limits for this so as to make effective utilization of time. In general, in queueing models, it is assumed that the customers arrive singly at service facility. But this assumption isviolated in many real life queueing situations.Control chart is a quality control technique evolved initially to monitor production processes. Montgomery (2010)proposed a number of applications of Shewhart control charts in assuring quality in manufacturing industries. Shore(2000) developed control chart for random queue length of M /M /S queueing model by considering the first threemoments and Poongodi and Muthulakshmi (2014) constructed the control chart for number of customers in thesystem of $M^{[X]}/M/1$ queueing system. Thus the analysis of time spent in the system by the control chart provides improvement of the performance of the system and hence customer satisfaction. In this research article is proposed to construct of control chart using process capability for average system length in $M^{[x]}/M/1$ Queuing system with numerical illustration is presented for the relevant study.

II. CONCEPTS AND TERMINOLOGIES

A.Arrival pattern

Arrival pattern describes the manner in which the units arrive and join thesystem. The source from which the units come may be finite or infinite.A unit may arrive either singly or in a group. The arrival pattern isoften measured in terms of the average number of arrivals per unit time.

B. Service pattern

Service pattern describes the manner in which the service is rendered to thearrivals. Customers may be served either singly or in batches. The time requiredfor serving a unit is called service time and the mean service rate is denoted by μ.The service pattern may be stationary or non-stationary with respect to time andstate dependent or independent with respect to number of customers waiting forservice.

C. Queue discipline

Queue discipline refers to the manner in which customers are selected for service from the queue. The most common disciplines based on the arrivals of customers into the system are first come first served (FCFS) and last come first served (LCFS). Customers may also be served randomly irrespective of their arrivals to the system called service in random order (SIRO).

D.Upper specification limit (USL)

It is the greatest amount specified by the producer for a process or product to have the acceptable performance.

E. Lower specification limit (LSL)

It is the smallest amount specified by the producer for a process or product to have the acceptable performance.

F.Tolerance level (TL)

It is a statistical interval within which, with some confidence level, a specified proportion of a sampled population falls. It is the difference between USL and LSL, TL = USL-LSL.

G.Process capability (C_P)

Process capability compares the output of an in-control process to the specification limits by using capability indices (Montgomery, 2010). The comparison is made by forming the ratio of the spread between the process specifications to the spread of the process values, as measured by 6 process standard deviation units.

i.e. $C_p = \frac{TL}{6\sigma} = \frac{USL - LSL}{6\sigma}$.

H. Average Run Length (ARL)

The average run length is the number of points that, on average, will be plotted on a control chart before an out of control condition is indicated (*www.micquality.com*).

If the process is in control:

$$ARL = \frac{1}{\alpha}$$

If the process is out of control:

$$ARL = \frac{1}{1-\beta}$$

where α is the probability of a Type I error and β the probability of a Type II error.

III. MODEL DESCRIPTION FOR $M^{[X]}/M/1$

Consider a single server queueing model in which the arrivals occur in batches according to Poisson process with rate$\lambda>0$. The batch size X is a random variable with $P(X=k)=C_k$, k=1, 2, … Customers are served one by one, the service time distribution is exponential with rate μ.

A.Steady stateequations

Let P_n be the probability that there are 'n' customers in the system.The steady state equations of this model are

$$0 = -(\lambda+\mu)P_n + \mu P_{n+1} + \lambda\sum_{k=1}^{n} P_{n-k}c_k, \qquad n \ge 1$$

$$o = -\lambda P_0 + \mu P_1$$

The above system of equations may be solved using generatingfunction approach.

Define the generating functions of the steady state probabilities $\{P_n\}$ and the batch size distribution $\{c_n\}$ respectively as

$$P(z) = \sum_{n=0}^{\infty} p_n z^n, \ |z| \le 1$$

$$C(z) = \sum_{n=1}^{\infty} c_n z^n, \ |z| \le 1$$

Multiplying the steady state equation by appropriate powers of z and summing, we get

$$0 = -\lambda\sum_{n=0}^{\infty} p_n z^n - \mu\sum_{n=1}^{\infty} p_n z^n + \frac{\mu}{z}\sum_{n=1}^{\infty} p_n z^n + \lambda\sum_{n=1}^{\infty}\sum_{k=1}^{n} p_{n-k}c_k z^n$$

Consider $\sum_{n=1}^{\infty}\sum_{k=1}^{n} p_{n-k}c_k z^n = \sum_{k=1}^{\infty} c_k z^k \sum_{n=k}^{\infty} P_{n-k} z^{n-k} = C(z)P(z)$

Using the above equations become

$$0 = -\lambda p(z) - \mu\left[P(z) - P_0\right] + \frac{\mu}{z}\left[P(z) - P_0\right] + \lambda C(z)P(z)$$

Solving for P(z), we get

$$P(z) = \frac{\mu P_0(1-z)}{\mu(1-z) - \lambda z\left[1 - C(z)\right]}, \ |z| \le 1$$

The generating function of the complementary batch size probabilities

$P(X > x) = 1 - C_x = \tilde{C}_x$ is given by

$$\tilde{C}(z) = \sum_{n-1}^{\infty} \tilde{c}_n z^n = \frac{1 - C(z)}{1 - z}$$

Taking $r = \frac{\lambda}{\mu}$, $P(z)$ yields

$$P(z) = \frac{P_0}{1 - rz\tilde{C}(z)}$$

Clearly, $\tilde{C}(1) = E(X)$ and $\tilde{C}'(1) = \frac{E\left(X\left(X-1\right)\right)}{2}$

Using the normalizing conditions, we obtain $P_0 = 1-\rho$, where $\rho = rE(X)$.

If Ns and Nq are number of customers in the system and the number of customers in the queue respectively then

$$N_s = \frac{\rho + rE(X^2)}{2(1-\rho)} \text{ and}$$

$$N_q = N_s - \rho$$

Assume that the number of customers in any arriving batch follows geometric distributionwith parameter α. Then the probability mass function of the batch size is

$$c_x = (1-\alpha)\alpha^{x-1},\ 0 < \alpha < 1.$$

$$\text{Then } C(z) = \frac{z(1-\alpha)}{1-\alpha z} \text{ and}$$

$$\text{E(X)=}\frac{1}{1-\alpha} \text{ with } \rho = \frac{r}{1-\alpha}$$

From the P(z) and C(z), we obtain

$$P(z) = (1-\rho)\left[\sum_{n=0}^{\infty}\left[\left(\alpha+(1-\alpha)\rho\right)z\right]^n - \sum_{n=0}^{\infty}\alpha\left[\alpha+(1-\alpha)\rho\right]^n z^{n+1}\right]$$

Comparison of like powers of z on both sides gives

$$P_n = \rho(1-\rho)(1-\alpha)\left[\alpha+(1-\alpha)\rho\right]^{n-1},\quad n > 0$$

B. Performance measures

Let N_s denote the number of customers in the system (both in queue and in service).Thenthe expected number of customers in the system is

$$E(N_s) = \frac{\rho}{(1-\rho)(1-\alpha)}$$

and the variance of the number of customers in the systemis

$$V(N_s) = \frac{\rho\left[1+\alpha(1-\rho)\right]}{(1-\rho)^2(1-\alpha)^2}$$

IV. METHODS AND MATERIALS

A.Control chart for average system length (N_s)

Shewhart (1931) type control charts are constructed by approximating the statistic under consideration by a normal distribution. The parameters of the control chart are given by

$$\text{UCL=E}(N_s)+3\sqrt{V(N_s)}$$

$$\text{CL=E}(N_s)$$

$$\text{LCL=E}(N_s)-3\sqrt{V(N_s)}$$

The parameters of the control chart (Poongodi and Muthulakshmi, 2014)for M/M/1 queueing model using $E(N_s)$ and $V(N_s)$, We get

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$$LCL = \frac{\rho}{(1-\alpha)(1-\rho)} - 3\frac{\sqrt{\rho[1+\alpha(1-\rho)]}}{(1-\alpha)(1-\rho)}$$

$$CL = \frac{\rho}{(1-\alpha)(1-\rho)}$$

$$UCL = \frac{\rho}{(1-\alpha)(1-\rho)} + 3\frac{\sqrt{\rho[1+\alpha(1-\rho)]}}{(1-\alpha)(1-\rho)}$$

B.Control chart for average system length (Ns) using process capability (Cp)

For a specified TL and C_p of the process (Radhakrishnan and Balamurugan, 2010), the value of σ (termed asσ_q) is calculated from $C_p = \left(TL/_{6\sigma}\right)$ using a computer program for various combinations of TL and C_p.

$$LCL_q = \frac{\rho}{(1-\alpha)(1-\rho)} - 3\sigma_q$$

$$CL_q = \frac{\rho}{(1-\alpha)(1-\rho)}$$

$$UCL_q = \frac{\rho}{(1-\alpha)(1-\rho)} + 3\sigma_q$$

V. ILLUSTRATION

Numerical analysis is carried out to analyze the performance of queueing system with reference to the parameters λ, μand α.

The following Table-I gives the traffic intensity and the control chart parameters for average system length forλ=2, μ=5 and α=0.11 to 0.20, we get

Table I: Control limits for average system length for λ=2, μ=5 and α=0.11 to 0.20

Arrival rate (λ)	Service rate (μ)	Number of customers in arriving batch (α)	Busy time (ρ)	Standard deviation (σ)	Control limits for $M^{[X]}/M/1$ Model			Control limits using process capability for $M^{[X]}/M/1$ Model (σ_q=0.049)	
					LCL	CL	UCL	LCL	UCL
2	5	0.11	0.449	1.409	-3.310	0.917	5.144	0.770	1.064
		0.12	0.455	1.450	-3.402	0.947	5.296	0.800	1.094
		0.13	0.460	1.492	-3.499	0.978	5.456	0.831	1.125
		0.14	0.465	1.537	-3.600	1.011	5.622	0.864	1.158
		0.15	0.471	1.584	-3.706	1.046	5.797	0.899	1.193
		0.16	0.476	1.633	-3.816	1.082	5.980	0.935	1.229
		0.17	0.482	1.684	-3.931	1.121	6.173	0.974	1.268
		0.18	0.488	1.738	-4.052	1.161	6.375	1.014	1.308
		0.19	0.494	1.794	-4.179	1.204	6.588	1.057	1.351
		0.2	0.500	1.854	-4.312	1.250	6.812	1.103	1.397

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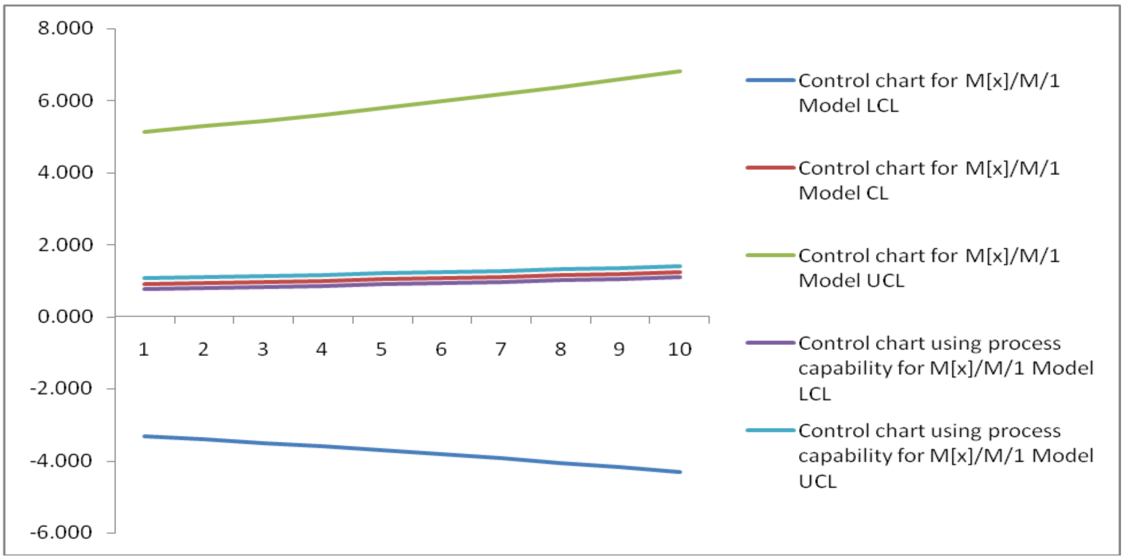

Figure I: Control limits for average system length for λ=2, μ=5 and α=0.11 to 0.20

From the above Table-I, it shows that when increasingthe probability of arriving customers in a batch with constant arrival rate (λ=2) and service rate (μ=5), then the average length of the system and the expected upper limits is also increases.This is shown in Figure-I.The control limits interval of 3σ using process capability is smaller than the control limits interval of Shewhart. It is clear that the existing approach is not in good quality as expected, accordingly a modification and improvement is needed in the queueing system.

The AverageRun Length (ARL) and the false alarm rate are obtained as follows:

Table II: Average Run Length (ARL) of control charts for $M^{[X]}/M/1$ Queueing system

σ	Shewhart control chart	Control chart using process capability
0.5	155.2242	143.5923
1	43.8947	41.1432
1.5	14.9677	14.1982
2	6.3030	6.0469
2.5	3.2411	3.1422
3	2.0000	1.9572

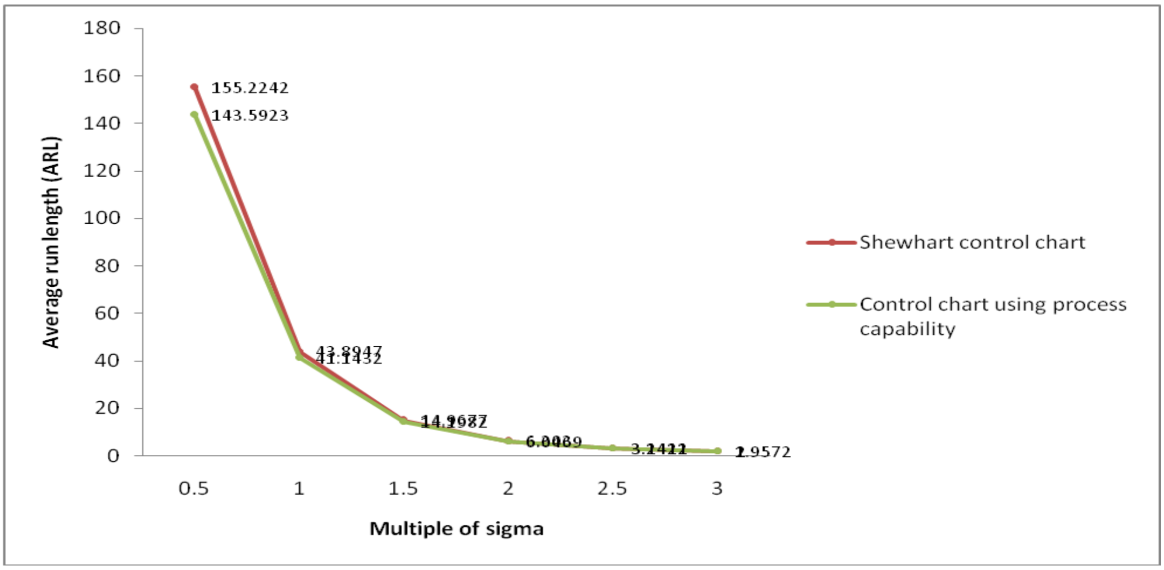

Figure II: Average run length (ARL) of control charts for $M^{[X]}/M/1$ Queueing system

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From the Figure-II, It is noticed that for the different values of**σ,**there is an decrease in the ARL of control chart using process capability for $M^{[X]}/M/1$ queueing model when compared with the Shewart control chart using process capability.Hence our model is more advisable and this can be observed in the above figure.

The following Table-III gives the traffic intensity and the control chart parameters foraverage system length forλ=2, μ=10 and α=0.11 to 0.20, we get

Table III: Control limits for average system length for λ=2, μ=10 and α=0.11 to 0.20

Arrival rate (λ)	Service rate (μ)	Number of customers in arriving batch (α)	Busy time (ρ)	Standard deviation (σ)	Control limits for $M^{[X]}/M/1$ Model			Control limits using process capability for $M^{[X]}/M/1$ Model (σ_q=0.020)	
					LCL	CL	UCL	LCL	UCL
2	10	0.11	0.225	0.716	-1.821	0.326	2.473	0.266	0.386
		0.12	0.227	0.733	-1.864	0.334	2.533	0.274	0.394
		0.13	0.230	0.751	-1.909	0.343	2.595	0.283	0.403
		0.14	0.233	0.769	-1.954	0.352	2.659	0.292	0.412
		0.15	0.235	0.788	-2.002	0.362	2.726	0.302	0.422
		0.16	0.238	0.808	-2.051	0.372	2.795	0.312	0.432
		0.17	0.241	0.828	-2.101	0.382	2.866	0.322	0.442
		0.18	0.244	0.849	-2.154	0.393	2.940	0.333	0.453
		0.19	0.247	0.871	-2.208	0.405	3.018	0.345	0.465
		0.20	0.250	0.894	-2.264	0.417	3.098	0.357	0.477

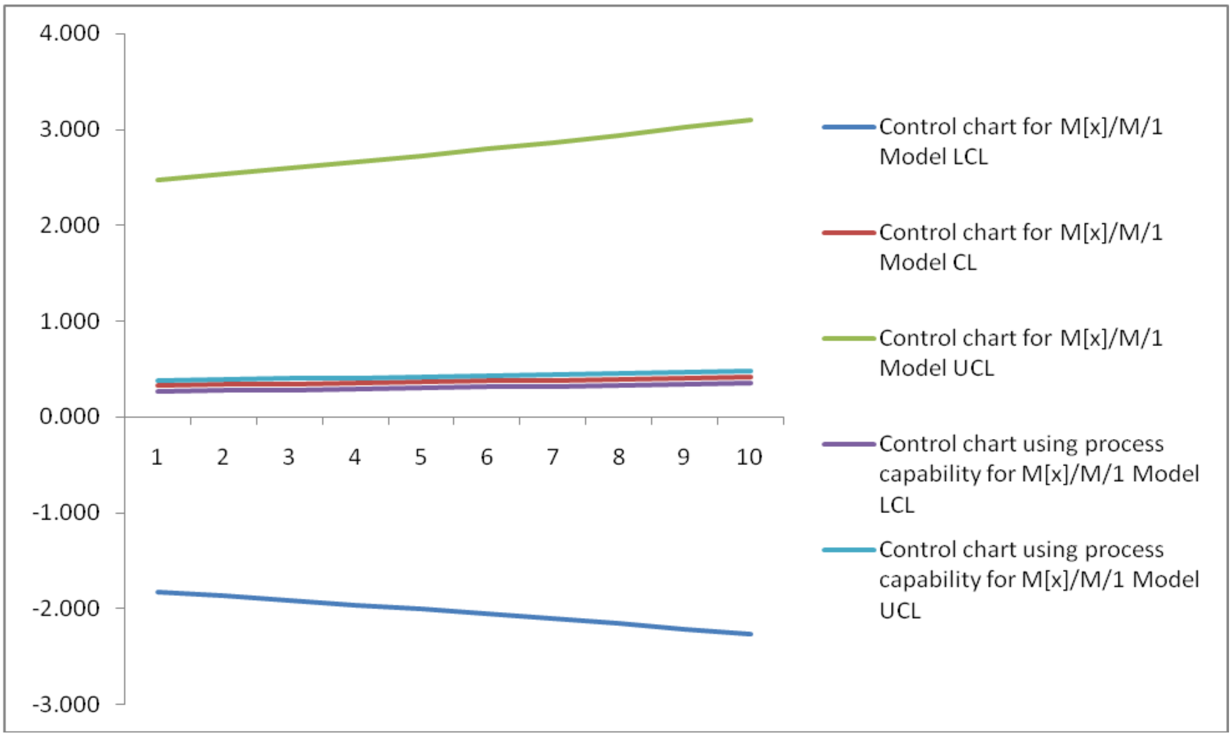

Figure III: Control limits for average system length for λ=2, μ=10 and α=0.11 to 0.20

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By observing the above Table-III,When there is an increasing in the number of customer in arriving batch with constant arrival(λ=2) and service rate (μ=10)there is an increases in the average system length and the expected upper limits .It is clear from the Figure-III that the control chart for 3σ using process capability performed better than the Shewhart chart.

The following Table-IV gives the traffic intensity and the control chart parameters for average system length forλ=3, μ=5 and α=0.11 to 0.20, we get

Table IV: Control limits for average system length for λ=3, μ=5 and α=0.11 to 0.20

Arrival rate (λ)	Service rate (μ)	Number of customers in arriving batch (α)	Busy time (ρ)	Standard deviation (σ)	Control limits for $M^{[X]}/M/1$ Model			Control limits using process capability for $M^{[X]}/M/1$ Model (σ_q=0.173)	
					LCL	CL	UCL	LCL	UCL
3	5	0.11	0.674	2.882	-6.320	2.325	10.969	1.806	2.844
		0.12	0.682	3.005	-6.579	2.435	11.449	1.916	2.954
		0.13	0.690	3.137	-6.857	2.554	11.966	2.035	3.073
		0.14	0.698	3.280	-7.156	2.683	12.523	2.164	3.202
		0.15	0.706	3.434	-7.478	2.824	13.126	2.305	3.343
		0.16	0.714	3.601	-7.827	2.976	13.779	2.457	3.495
		0.17	0.723	3.783	-8.205	3.143	14.491	2.624	3.662
		0.18	0.732	3.981	-8.617	3.326	15.269	2.807	3.845
		0.19	0.741	4.198	-9.067	3.527	16.122	3.008	4.046
		0.20	0.750	4.437	-9.561	3.750	17.061	3.231	4.269

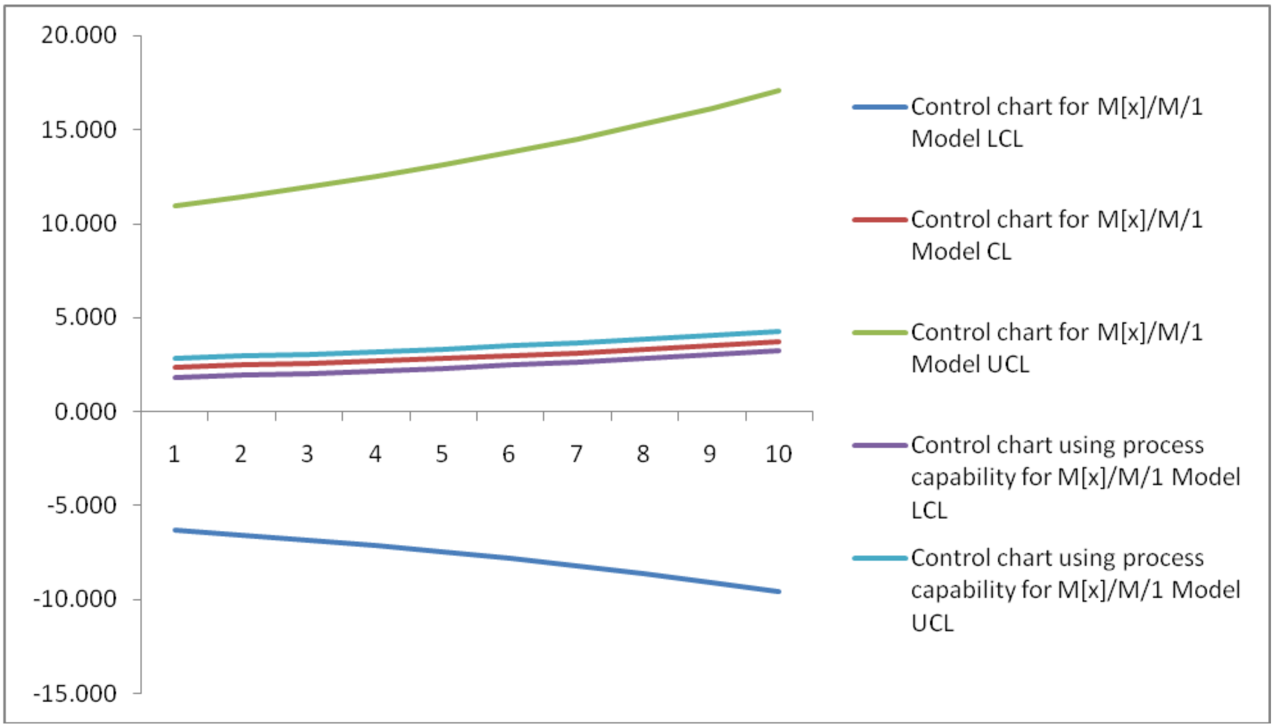

Figure IV: Control limits for average system length for λ=3, μ=5 and α=0.11 to 0.20

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Based on the numerical result obtained in Table-IV, for a constant value ofλ=3,μ=5 and an increase in the number of customer arrival to a batch, we findan increases in the average system length and the expected upper limits. The Figure-IV reveals that the control limits interval of 3σ using process capability is smaller than the control limits interval of Shewhart.

The following Table-V gives the traffic intensity and the control chart parameters for average system length forλ=3, μ=10 and α=0.11 to 0.20, we get

Table V: Control limits for average system length for λ=3, μ=10 and α=0.11 to 0.20

Arrival rate (λ)	Service rate (μ)	Number of customers in arriving batch (α)	Busy time (ρ)	Standard deviation (σ)	Control limits for $M^{[X]}/M/1$ Model			Control limits using process capability for $M^{[X]}/M/1$ Model (σ_q=0.031)	
					LCL	CL	UCL	LCL	UCL
3	10	0.11	0.337	1.019	-2.487	0.571	3.629	0.478	0.664
		0.12	0.341	1.046	-2.549	0.588	3.725	0.495	0.681
		0.13	0.345	1.073	-2.615	0.605	3.825	0.512	0.698
		0.14	0.349	1.102	-2.682	0.623	3.928	0.530	0.716
		0.15	0.353	1.131	-2.752	0.642	4.036	0.549	0.735
		0.16	0.357	1.162	-2.825	0.661	4.148	0.568	0.754
		0.17	0.361	1.194	-2.901	0.682	4.265	0.589	0.775
		0.18	0.366	1.228	-2.980	0.704	4.387	0.611	0.797
		0.19	0.370	1.263	-3.062	0.726	4.514	0.633	0.819
		0.20	0.375	1.299	-3.147	0.750	4.647	0.657	0.843

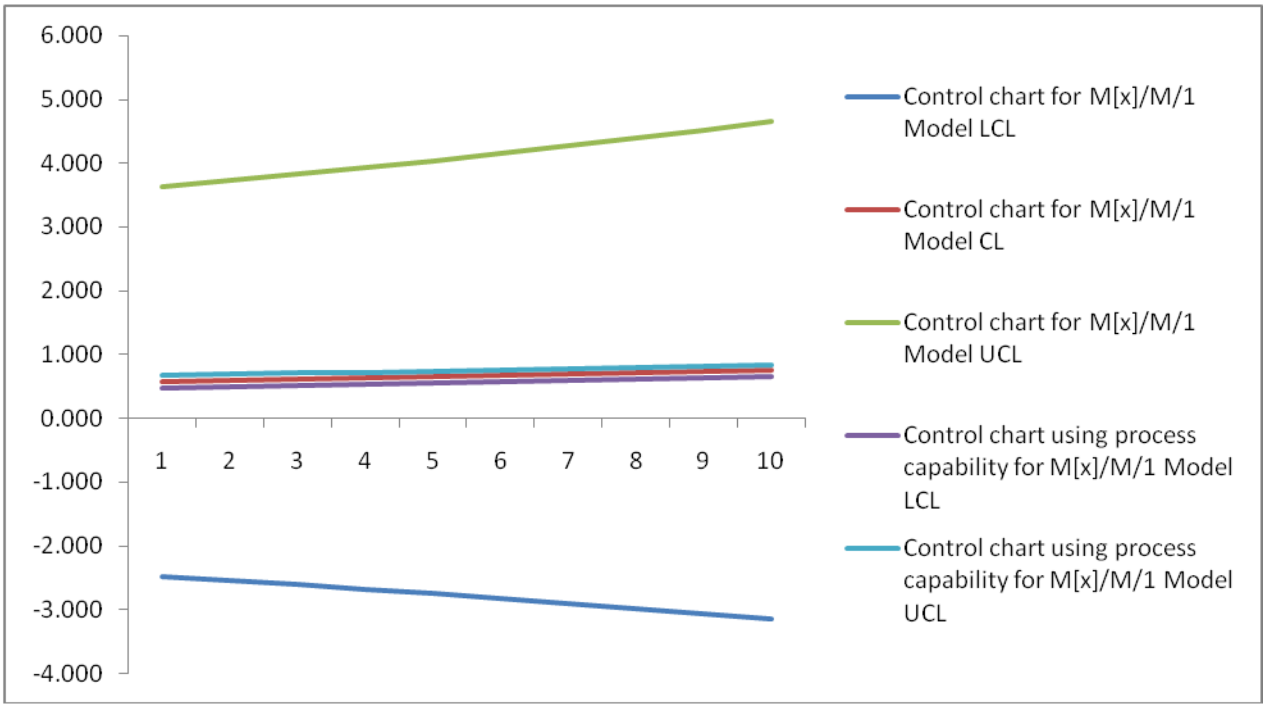

Figure V: Control limits for average system length for λ=3, μ=10 and α=0.11 to 0.20

From the above Table-V and Figure-V, there is an increasing trend in CL and UCL when increasein the number of customers in arriving batch with constant arrival λ=3 and service rateμ=10.The Figure-V, shows the improvement in the quality of waiting time in the control chart using process capability.

The following Table-VI gives the traffic intensity and the control chart parameters for average system length forλ=5, α=0.15 and μ=15 to 60 we get

Table VI: Control limits for average system length for λ=5, α=0.15 and μ=15 to 60

Arrival rate (λ)	Number of customers in arriving batch (α)	Service rate (μ)	Busy time (ρ)	Standard deviation (σ)	Control limits for $M^{[X]}/M/1$ Model			Control limits using process capability for $M^{[X]}/M/1$ Model (σ_q=0.092)	
					LCL	CL	UCL	LCL	UCL
5	0.15	15	0.392	1.266	-3.039	0.759	4.557	0.483	1.035
		20	0.294	0.951	-2.361	0.490	3.342	0.214	0.766
		25	0.235	0.788	-2.002	0.362	2.726	0.086	0.638
		30	0.196	0.686	-1.771	0.287	2.345	0.011	0.563
		35	0.168	0.615	-1.607	0.238	2.082	-0.038	0.514
		40	0.147	0.562	-1.482	0.203	1.888	-0.073	0.479
		45	0.131	0.520	-1.384	0.177	1.738	-0.099	0.453
		50	0.118	0.487	-1.303	0.157	1.617	-0.119	0.433
		55	0.107	0.459	-1.235	0.141	1.517	-0.135	0.417
		60	0.098	0.435	-1.178	0.128	1.433	-0.148	0.404

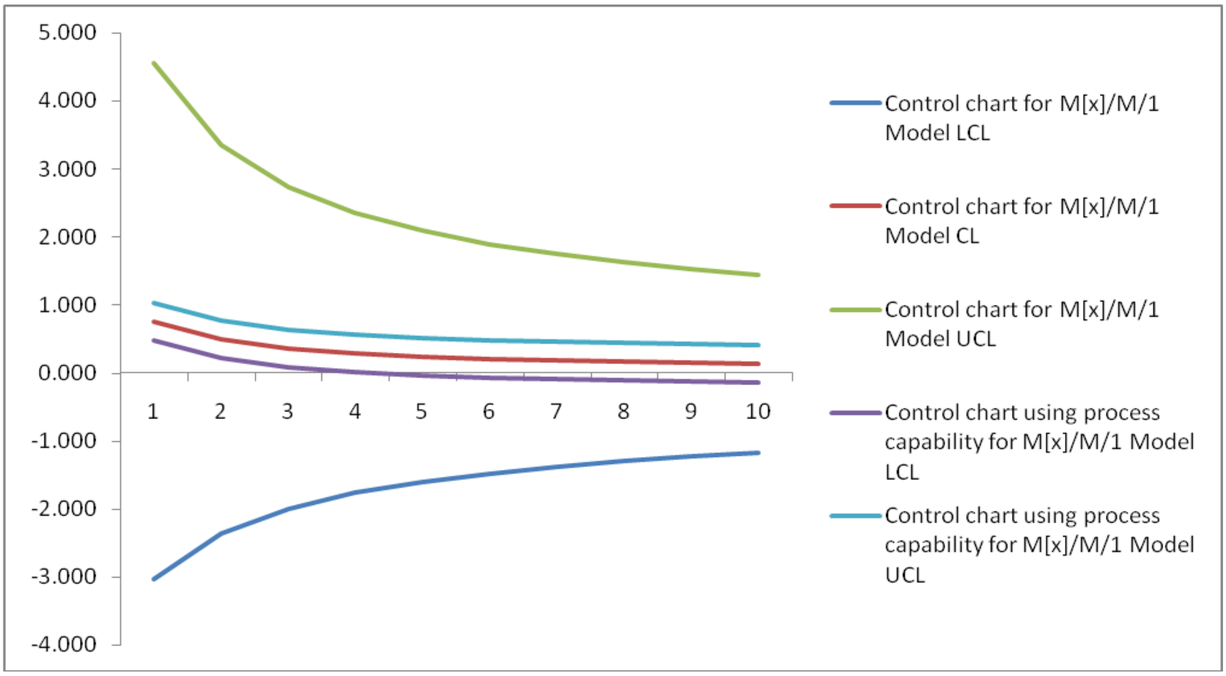

Figure VI: Control limits for average system length for λ=5, α=0.15 and μ=15 to 60

From the Table-VI,

(i) While increasing the service rate with constant arrival rate (λ=5) and number of customers in arriving batch(α=0.15)there isa decrease in the average system length and UCL.

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(ii) The Figure-VI, shows the variation between control limits of Shewhart and control limit using process capability and proves thatt he control chart using process capability is more potential.

The following Table-VII gives the traffic intensity and the control chart parameters for average system length forμ=20, α =0.15 and λ=6 to 15 we get

Table VII: Control limits for average system length for μ=20, α=0.15 and λ=6 to 15

Service rate (μ)	Number of customers in arriving batch (α)	Arrival rate (λ)	Busy time (ρ)	Standard deviation (σ)	Control limits for $M^{[X]}/M/1$ Model			Control limits using process capability for $M^{[X]}/M/1$ Model (σ_q=0.092)	
					LCL	CL	UCL	LCL	UCL
20	0.15	6	0.353	1.131	-2.752	0.642	4.036	-2.139	3.423
		7	0.412	1.339	-3.193	0.824	4.840	-1.957	3.605
		8	0.471	1.584	-3.706	1.046	5.797	-1.735	3.827
		9	0.529	1.882	-4.323	1.324	6.970	-1.457	4.105
		10	0.588	2.258	-5.093	1.681	8.455	-1.100	4.462
		11	0.647	2.751	-6.097	2.157	10.411	-0.624	4.938
		12	0.706	3.434	-7.478	2.824	13.126	0.043	5.605
		13	0.765	4.449	-9.523	3.824	17.170	1.043	6.605
		14	0.824	6.129	-12.898	5.490	23.879	2.709	8.271
		15	0.882	9.476	-19.604	8.824	37.251	6.043	11.605

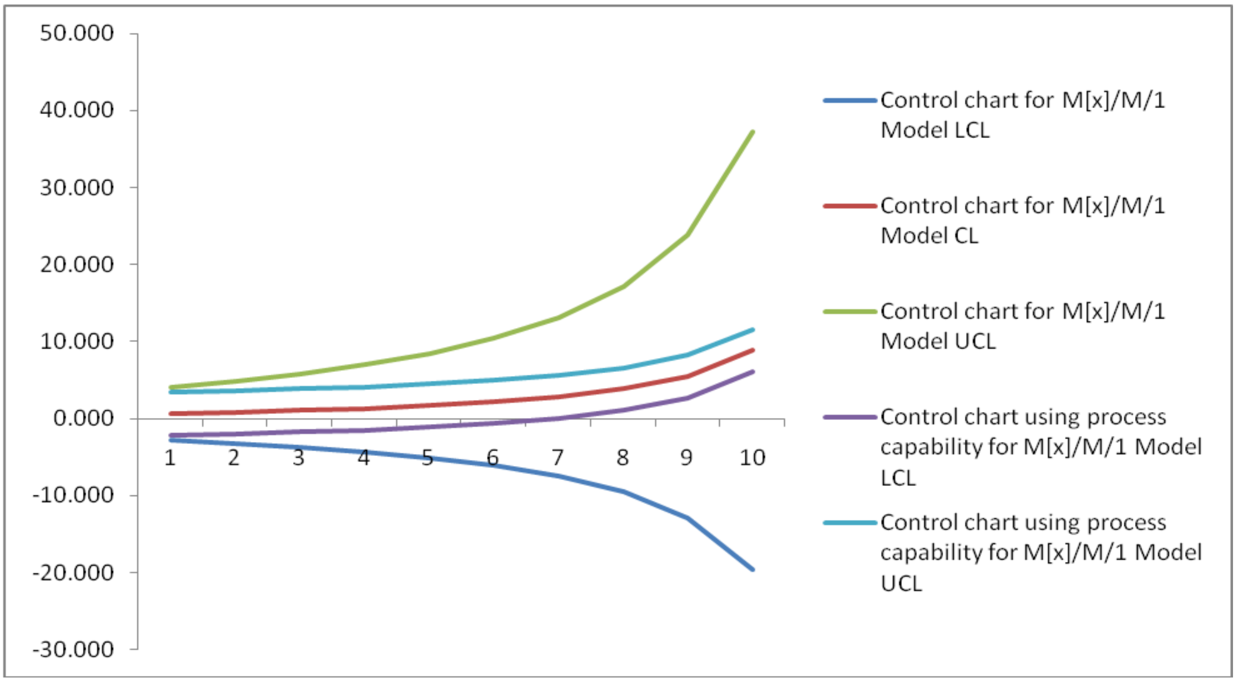

Figure VII: Control limits for average system length for μ=20, α=0.15 and λ=6 to 15

From the above Table-VII, increase in the arrival rate with constant service rate (μ=20) and number of customers in arriving batch (α=0.15) increases the CL and UCL.

The Figure-VII, compares the control charts of CL and UCL.Control chart using process capability gives the right warning to take necessary action.

VI. CONCLUSION

In this article we have proposed a comparative study between Shewart control chart and control chart using process capability for average system length that includes different arrival rate, service rate and the customers' arrival to the batch to avoid the customers excessive waiting time in the queue. It has been presented in the form of average run length also. The obtained results prove that control chart using process capability is more potential than the comparative one, asit alert us even before the exiting chart. Hence, the proposed chart isrecommended andleads to take necessary actions to reduce the waiting time.

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